

HELSON'S PROBLEM FOR SUMS OF A RANDOM MULTIPLICATIVE FUNCTION

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ABSTRACT. We consider the random functions $S_N(z) := \sum_{n=1}^N z(n)$, where $z(n)$ is the completely multiplicative random function generated by independent Steinhaus variables $z(p)$. It is shown that $\mathbb{E}|S_N| \gg \sqrt{N}(\log N)^{-0.05616}$ and that $(\mathbb{E}|S_N|^q)^{1/q} \gg_q \sqrt{N}(\log N)^{-0.07672}$ for all $q > 0$.

1. INTRODUCTION

This paper deals with the following

Question. *Do there exist absolute constants $c > 0$, $0 < \lambda < 1$ such that for every positive integer N and every interval I whose length exceeds some number depending on N , we have*

$$\left| \sum_{n=1}^N n^{-it} \right| \geq c\sqrt{N}$$

on a subset of I of measure larger than $\lambda|I|$?

We do not know the answer and can only conclude from our main result that we have, for every $\varepsilon > 0$ and suitable $c = c(\varepsilon)$,

$$(1) \quad \left| \sum_{n=1}^N n^{-it} \right| \geq c\sqrt{N}(\log N)^{-0.07672}$$

on a subset of measure $(\log N)^{-\varepsilon}|I|$ of every sufficiently large interval I .

Our question fits into the following general framework. We begin by associating with every prime p a random variable $X(p)$ with mean 0 and variance 1, and we assume that these variables are independent and identically distributed. We then define $X(n)$ by requiring it to be a completely multiplicative function for every point in our probability space. Now suppose that $a(n)$ is an arithmetic function which is either 0 or 1 for every n . We refer to the sequence

$$C_N(X) := \sum_{n=1}^N a(n)X(n)$$

as the arithmetic chaos associated with X and $a(n)$.

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Our question concerns the case when $X(p)$ are independent Steinhaus variables $z(p)$, i.e. the random variable $z(p)$ is equidistributed on the unit circle. When $a(n) \equiv 1$, we refer to the resulting sequence

$$S_N(z) := \sum_{n=1}^N z(n)$$

as arithmetic Steinhaus chaos. The relation between our question and arithmetic Steinhaus chaos is given by the well-known norm identity

$$(2) \quad \mathbb{E}(|S_N|^q) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left| \sum_{n=1}^N n^{-it} \right|^q dt,$$

valid for all $q > 0$ (see [12, Section 3]).

The point of departure for our research is Helson's last paper [10] in which he conjectured that $\mathbb{E}(|S_N|) = o(\sqrt{N})$ when $N \rightarrow \infty$. This means that Helson anticipated that our question has a negative answer. Using an inequality from another paper of Helson [9], we get immediately that

$$(3) \quad \mathbb{E}(|S_N|) \gg \sqrt{N}(\log N)^{-1/4}.$$

Our attempt to settle Helson's conjecture has resulted in a reduction from $1/4$ to 0.05616 in the exponent of the logarithmic factor in (3). We note in passing that the problem leading Helson to his conjecture was solved in [11] avoiding the use of the random functions S_N .

To get a picture of what our work is about, it is instructive to return for a moment to a general arithmetic chaos $C := (C_N(X))$. To this end, let us assume that $X(p)$ is such that the moments

$$\|C_N\|_q^q := \mathbb{E}(|C_N|^q)$$

are well defined for all $q > 0$. We declare the number

$$q(C) := \inf \left\{ q > 0 : \limsup_{N \rightarrow \infty} \|C_N\|_{q+\varepsilon} / \|C_N\|_q = \infty \text{ for every } \varepsilon > 0 \right\}$$

to be the critical exponent of C , setting $q(C) = \infty$ should the set on the right-hand side be empty. A problem closely related to Helson's conjecture is that of computing the critical exponent of a given arithmetic chaos. We observe that $q(C) \geq 2$ is equivalent to the statement that there exist absolute constants $c > 0$, $0 < \lambda < 1$ such that

$$\mathbb{P}(|C_N| \geq c\sqrt{N}) \geq \lambda$$

holds for all N , cf. our question. In our case, the critical exponent is strictly smaller than 4, and then a serious obstacle for saying much more is that only even moments are accessible by direct methods.

We will prove the following result about arithmetic Steinhaus chaos.

Theorem 1. *We have*

$$(4) \quad \|S_N\|_q \gg_q \sqrt{N}(\log N)^{-0.07672}$$

for all $q > 0$.

This estimate is of course of interest only for small q ; our method allows us to improve (4) for each individual $0 < q < 2$ as will be demonstrated in the last section of the paper. In the range $q > 2$, we note that the L^4 norm has an interesting number theoretic interpretation and has been estimated with high precision [1]:

$$\|S_N\|_4^4 = \frac{12}{\pi^2} N^2 \log N + cN^2 + O(N^{19/13}(\log N)^{7/13})$$

with c a certain number theoretic constant. This means in particular that the critical exponent of arithmetic Steinhaus chaos $S := (S_N)$ satisfies $q(S) < 4$. We mention without proof that, applying the Hardy–Littlewood inequality from [2] to S_N^2 , we have been able to verify that in fact $q(S) \leq 8/3$. A further elaboration of our methods could probably lower this estimate slightly, but this would not alter the main conclusion that it remains unknown whether $q(S)$ is positive.

Before turning to the proof of Theorem 1, we mention the following simple fact: There exists a constant $c < 1$ such that $\|S_N\|_1 \leq c\|S_N\|_2$ when $N \geq 2$. To see this, we apply the Cauchy–Schwarz inequality to the product of $(1 - \varepsilon z(2))S_N$ and $(1 - \varepsilon z(2))^{-1}$ to obtain

$$\|S_N\|_1^2 \leq \frac{1}{(1 - \varepsilon^2)} \cdot ((1 - \varepsilon)^2 [N/2] + (1 + \varepsilon^2) [(N + 1)/2]) \leq \frac{N - (\varepsilon - \varepsilon^2)(N - 1)}{1 - \varepsilon^2}$$

for every $0 < \varepsilon < 1$. Choosing a suitable small ε , we obtain the desired constant $c < 1$.

2. PROOF OF THEOREM 1

Our proof starts from a decomposition of S_N into a sum of homogeneous polynomials. To this end, we set

$$E_{N,m} := \{n \leq N : \Omega(n) = m\},$$

where $\Omega(n)$ is the number of prime factors of n , counting multiplicities. Correspondingly, we introduce the homogeneous polynomials

$$S_{N,m}(z) := \sum_{n \in E_{N,m}} z(n)$$

so that we may write

$$(5) \quad S_N(z) = \sum_{m \leq (\log N)/\log 2} S_{N,m}(z).$$

We need two lemmas. The first is a well-known estimate of Sathe; the standard reference for this result is Selberg's paper [13]. To formulate this lemma, we introduce the function

$$\Phi(z) := \frac{1}{\Gamma(z+1)} \prod_p (1 - 1/p)^z (1 - z/p)^{-1},$$

where the product runs over all prime numbers p . This function is meromorphic in \mathbb{C} with simple poles at the primes and zeros at the negative integers.

Lemma 2. *When $N \geq 3$ and $1 \leq m \leq (2 - \varepsilon) \log \log N$ for $0 < \varepsilon < 1$, we have*

$$|E_{N,m}| = \frac{N}{\log N} \Phi\left(\frac{m}{\log \log N}\right) \frac{(\log \log N)^{m-1}}{(m-1)!} \left(1 + O\left(\frac{1}{\log \log N}\right)\right),$$

where the implied constant in the error term only depends on ε .

The second lemma is a general statement about the decomposition of a holomorphic function into a sum of homogeneous polynomials. For simplicity, we consider only an arbitrary holomorphic polynomial $P(z)$ in d complex variables $z = (z_1, \dots, z_d)$. Such a polynomial has a unique decomposition

$$P(z) = \sum_{m=0}^k P_m(z),$$

where k is the degree of P and

$$P_m(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$$

is a homogeneous polynomial of degree m . Here we use standard multi-index notation, which means that $\alpha = (\alpha_1, \dots, \alpha_d)$, where $\alpha_1, \dots, \alpha_d$ are nonnegative integers,

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d},$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. At this point, the reader should recognize that if we represent an arbitrary integer $n \leq N$ by its prime factorization $p_1^{\alpha_1} \cdots p_d^{\alpha_d}$ (here $d = \pi(N)$) and set $\alpha(n) = (\alpha_1, \dots, \alpha_d)$, then we may write

$$S_N(z) = \sum_{n=1}^N z^{\alpha(n)}.$$

Hence, as already pointed out, (5) is the decomposition of S_N into a sum of homogeneous polynomials, and we also see that $|\alpha(n)| = \Omega(n)$.

We let μ_d denote normalized Lebesgue measure on \mathbb{T}^d and define

$$\|P\|_q^q := \int_{\mathbb{T}^d} |P(z)|^q d\mu_d(z)$$

for every $q > 0$. The variables z_1, \dots, z_d can be viewed as independent Steinhaus variables so that $\|S_N\|_q$ has the same meaning as before.

Lemma 3. *There exists an absolute constant C , independent of d , such that*

$$\|P_m\|_q \leq \begin{cases} \|P\|_q, & q \geq 1 \\ Cm^{1/q-1} \|P\|_q, & 0 < q < 1 \end{cases}$$

holds for every holomorphic polynomial P of d complex variables.

Proof. We introduce the transformation $z_w = (wz_1, \dots, wz_d)$, where w is a point on the unit circle \mathbb{T} . We may then write

$$P(z_w) = \sum_{m=0}^k P_m(z) w^m.$$

It follows that we may consider the polynomials $P_m(z)$ as the coefficients of a polynomial in one complex variable. Then a classical coefficient estimate (see [4, p. 98]) shows that

$$|P_m(z)|^q \leq \begin{cases} \int_{\mathbb{T}} |P_m(z_w)|^q d\mu_1(w), & q \geq 1 \\ Cm^{1-q} \int_{\mathbb{T}} |P_m(z_w)|^q d\mu_1(w), & 0 < q < 1. \end{cases}$$

Integrating this inequality over \mathbb{T}^d with respect to $d\mu_d(z)$ and using Fubini's theorem, we obtain the desired estimate. \square

We now turn to the proof of Theorem 1. The idea of the proof can be related to an interesting study of Harper [6] from which it can be deduced that, asymptotically, the square-free part of the homogeneous polynomial $S_{N,m}/\sqrt{N}$ has a Gaussian distribution when $m = o(\log \log N)$. When $m = \beta \log \log N$ for β bounded away from 0, this is no longer so, but what we will use, is a much weaker statement: When β is small enough, the L^2 and L^4 norms are comparable. The proof will consist in identifying for which β this holds.

To this end, we first observe that

$$(6) \quad \|S_{N,m}\|_2^2 = |E_{N,m}|.$$

To estimate $\|S_{N,m}\|_4^4$, we begin by noting that

$$|S_{N,m}|^2 = |E_{N,m}| + \sum_{k=0}^m \sum_{a,b \in E_{N,k}, (a,b)=1} |E_{N/\max(a,b),m-k}| z(a) \overline{z(b)}.$$

Squaring this expression and taking expectation, we obtain

$$(7) \quad \begin{aligned} \|S_{N,m}\|_4^4 &= |E_{N,m}|^2 + 2 \sum_{k=0}^m \sum_{a,b \in E_{N,k}, (a,b)=1, a < b} |E_{N/b,m-k}|^2 \\ &\leq |E_{N,m}|^2 + 2 \sum_{k=0}^m \sum_{b \in E_{N,k}} |E_{b,k}| \cdot |E_{N/b,m-k}|^2 \\ &\leq 5|E_{N,m}|^2 + \sum_{k=1}^{m-1} \sum_{b \in E_{N,k}} |E_{b,k}| \cdot |E_{N/b,m-k}|^2. \end{aligned}$$

Here we used that, plainly,

$$\sum_{b \in E_{N,0}} |E_{N/b,m}|^2 = |E_{N,m}|^2 \quad \text{and} \quad \sum_{b \in E_{N,m}} |E_{b,m}| \leq |E_{N,m}|^2.$$

To estimate the sum over b in (7), we begin by observing that Lemma 2 implies that

$$(8) \quad \begin{aligned} |E_{N/b,m-k}| &\ll b^{-1} |E_{N,m-k}|, \quad b \leq \sqrt{N}, \\ |E_{b,k}| &\ll b N^{-1} |E_{N,k}|, \quad \sqrt{N} < b \leq N. \end{aligned}$$

We split correspondingly the sum into two parts:

$$(9) \quad \sum_{b \in E_{N,k}} |E_{b,k}| \cdot |E_{N/b,m-k}|^2 \ll |E_{N,m-k}|^2 \sum_{b \in E_{\sqrt{N},k}} b^{-2} |E_{b,k}| + |E_{N,k}| \sum_{b \in E_{N,k} \setminus E_{\sqrt{N},k}} b N^{-1} |E_{N/b,m-k}|^2.$$

To deal with the first of the two sums in (9), we begin by using Lemma 2 so that we get smooth terms in the sum:

$$\sum_{b \in E_{\sqrt{N},k}} b^{-2} |E_{b,k}| \ll \sum_{b \in E_{\sqrt{N},k} \setminus \{1,2\}} \frac{(\log \log b)^{k-1}}{b(\log b)(k-1)!} = \sum_{2 < b \leq \sqrt{N}} g(b) \frac{(\log \log b)^{k-1}}{b(\log b)(k-1)!},$$

where $g(n)$ is the characteristic function of the set $E_{\sqrt{N},k}$. We apply Abel's summation formula to the latter sum and obtain, using also Lemma 2,

$$\begin{aligned}
 \sum_{b \in E_{\sqrt{N},k}} b^{-2} |E_{b,k}| &\ll |E_{\sqrt{N},k}| N^{-1/2} \frac{(\log \log N)^{k-1}}{(\log N)(k-1)!} + \int_3^N |E_{x,k}| \frac{(\log \log x)^{k-1}}{x^2 (\log x)(k-1)!} dx \\
 &\ll \frac{(\log \log N)^{2(k-1)}}{(\log N)^2 ((k-1)!)^2} + \int_3^N \frac{(\log \log x)^{2(k-1)}}{x (\log x)^2 ((k-1)!)^2} dx \\
 (10) \quad &\ll \frac{1}{((k-1)!)^2} \int_0^\infty y^{2(k-1)} e^{-y} dy = \frac{(2k-2)!}{((k-1)!)^2} \ll \frac{2^{2k}}{\sqrt{k}}.
 \end{aligned}$$

Arguing in a similar fashion, using Abel's summation formula and again (8), we get

$$\begin{aligned}
 \sum_{b \in E_{N,k} \setminus E_{\sqrt{N},k}} b N^{-1} |E_{N/b, m-k}|^2 &\ll N \int_{\sqrt{N}}^{N/3} |E_{x,k}| \frac{\left(\log \log \frac{N}{x}\right)^{2(m-k-1)}}{x^2 \left(\log \frac{N}{x}\right)^2 ((m-k-1)!)^2} dx \\
 &\ll \frac{|E_{N,k}|}{((m-k-1)!)^2} \int_0^\infty y^{2(m-k-1)} e^{-y} dy \\
 (11) \quad &\ll |E_{N,k}| \cdot \frac{2^{2(m-k)}}{\sqrt{m-k}}.
 \end{aligned}$$

Inserting (10) and (11) into (9), we obtain

$$\sum_{b \in E_{N,k}} |E_{b,k}| \cdot |E_{N/b, m-k}|^2 \ll |E_{N, m-k}|^2 \cdot \frac{2^{2k}}{\sqrt{k}} + |E_{N,k}|^2 \cdot \frac{2^{2(m-k)}}{\sqrt{m-k}}.$$

Returning to (7) and using (6), we therefore find that

$$\|S_{N,m}\|_4^4 \ll \|S_{N,m}\|_2^4 \left(1 + \sum_{k=1}^{m-1} \frac{|E_{N, m-k}|^2}{|E_{N,m}|^2} \cdot \frac{2^{2k}}{\sqrt{k}}\right).$$

Applying again Lemma 2, we get

$$\begin{aligned}
 \sum_{k=1}^{m-1} \frac{|E_{N, m-k}|^2}{|E_{N,m}|^2} \cdot \frac{2^{2k}}{\sqrt{k}} &\ll \sum_{k=1}^{m-1} \frac{1}{\sqrt{k}} \cdot \left(\frac{(m-1)!}{(m-k-1)!}\right)^2 \cdot \left(\frac{2}{\log \log N}\right)^{2k} \\
 (12) \quad &\ll \sum_{k=1}^{m-1} \frac{1}{\sqrt{k}} \cdot e^{-\frac{k^2}{m}} \cdot \left(\frac{2m}{\log \log N}\right)^{2k}.
 \end{aligned}$$

It follows that the two norms are comparable whenever $m = \frac{e^{-\varepsilon}}{2} \log \log N$ for $\varepsilon > 0$, in which case Hölder's inequality yields

$$(13) \quad \|S_{N,m}\|_2 \ll_\varepsilon \|S_{N,m}\|_q$$

for $0 < q < 2$. By (6), Lemma 2, and Stirling's formula, we have

$$(14) \quad \|S_{N,m}\|_2 = |E_{N,m}|^{1/2} \asymp \sqrt{N} (\log N)^{-\delta(\varepsilon)} m^{-1/4}$$

when $m \leq \frac{e^{-\varepsilon}}{2} \log \log N$, where

$$\delta(\varepsilon) := (2 - e^{-\varepsilon}(1 + \log 2 + \varepsilon))/4 = (1 - \log 2)/4 + O(\varepsilon)$$

when $\varepsilon \rightarrow 0$. Combining this with (13) and applying Lemma 3, we infer that

$$\sqrt{N}(\log N)^{-\delta(\varepsilon)}(\log \log N)^{-1/4} \ll \|S_{N,m}\|_q \ll (\log \log N)^{\max(1/q-1,0)} \|S_N\|_q$$

when $0 < q < 2$. Theorem 1 follows since $(1 - \log 2)/4 < 0.07672$.

3. CONCLUDING REMARKS

1. We will now deduce (1) from Theorem 1. Since $t \mapsto \sum_{n=1}^N n^{-it}$ is an almost periodic function, it suffices to consider the interval $I = [0, T]$ for some large T . Moreover, by (2), it amounts to the same to estimate the measure of the subset

$$\mathcal{E} := \left\{ z : |S_N(z)| \geq c\sqrt{N}(\log N)^{-0.07672} \right\}$$

of $\mathbb{T}^{\pi(N)}$ for a suitable c depending on ε . We find that

$$\begin{aligned} \|S_N\|_q^q &\leq c^q N^{q/2} (\log N)^{-0.07672q} + \int_{\mathcal{E}} |S_N(z)|^q d\mu_{\pi(N)}(z) \\ &\leq c^q N^{q/2} (\log N)^{-0.07672q} + \|S_N\|_2^q |\mathcal{E}|^{1-q/2}, \end{aligned}$$

where we in the last step used Hölder's inequality. Using Theorem 1 to estimate $\|S_N\|_q^q$ from below and recalling that $\|S_N\|_2 = \sqrt{N}$, we therefore get

$$(15) \quad \kappa_q (\log N)^{-0.07672q} \leq c^q (\log N)^{-0.07672q} + |\mathcal{E}|^{1-q/2},$$

where κ_q is a constant depending on q . Given $\varepsilon > 0$, we now choose q such that $\varepsilon = 0.07672q/(1 - q/2)$ and $c^q = \kappa_q/2$. Then (15) yields

$$|\mathcal{E}| \geq (\kappa_q/2)^{(1-q/2)^{-1}} (\log N)^{-\varepsilon}.$$

2. We may improve (4) in the following way. If $m = \frac{e^y}{2} \log \log N$ with $y > 0$, then we see from (12) that

$$\sum_{k=1}^{m-1} \frac{|E_{N,m-k}|^2}{|E_{N,m}|^2} \cdot \frac{2^{2k}}{\sqrt{k}} \ll_y e^{my^2}$$

which in turn implies that $\|S_{N,m}\|_2 / \|S_{N,m}\|_4 \gg_y e^{-my^2/4}$. By Hölder's inequality,

$$\|S_{N,m}\|_2 \leq \|S_{N,m}\|_q^{\frac{q}{4-q}} \|S_{N,m}\|_4^{\frac{4-2q}{4-q}},$$

and we therefore get

$$\|S_{N,m}\|_q \gg_y \|S_{N,m}\|_2 e^{-my^2(2/q-1)/2} = \|S_{N,m}\|_2 (\log N)^{-e^y y^2(2/q-1)/4}.$$

We also observe that (14) now takes the form

$$\|S_{N,m}\|_2 = |E_{N,m}|^{1/2} \asymp \sqrt{N}(\log N)^{(e^y(1+\log 2-y)-2)/4} m^{-1/4}.$$

We find that the exponent of $\log N$ in the lower bound for $\|S_{N,m}\|_q$ becomes optimal if we choose y as the positive solution to the quadratic equation $(2/q-1)y^2 + (4/q-1)y - \log 2 = 0$; when $q = 1$, we get for instance $y = 0.21556\dots$ and hence, after a numerical calculation,

$$\|S_N\|_1 \gg \sqrt{N}(\log N)^{-0.05616}.$$

3. Our proof shows that we essentially need $\beta \leq 1/2$ for the projection

$$P_\beta S_N := \sum_{m \leq \beta \log \log N} S_{N,m}$$

to have comparable L^2 and L^4 norms. To use our method of proof to show that Helson's conjecture fails, it would suffice to know that the projection $P_1 S_N$ has comparable L^2 and L^q norms for some $q > 2$, because in that case $\|P_1 S_N\|_2 \geq (1 + o(1))\sqrt{N}$. However, we see no reason to expect that such a q exists.

4. A careful examination of our proof, including a detailed estimation of the last sum in (12), shows that

$$(16) \quad \|S_{N,m}\|_4 / \|S_{N,m}\|_2 \asymp (\log \log N)^{1/16}$$

when $m = \frac{1}{2} \log \log N + O(\sqrt{\log \log N})$. This means that $\frac{1}{2} \log \log N$ is indeed the critical degree of homogeneity and, moreover, that the two norms fail to be comparable in the limiting case.

5. Helson's problem makes sense for other distributions; an interesting case is when $X(p)$ are independent Rademacher functions $\epsilon(p)$ taking values $+1$ and -1 each with probability $1/2$. If we set $a(n) = |\mu(n)|$ (here $\mu(n)$ is the Möbius function), then we obtain arithmetic Rademacher chaos:

$$R_N(\epsilon) := \sum_{n=1}^N |\mu(n)| \epsilon(n).$$

Rademacher chaos was first considered by Wintner [14] and has been studied by many authors, see e.g. [5, 6]. Here it is of interest to note that Chatterjee and Soundararajan showed that $R_{N+y} - R_N$ is approximately Gaussian when $y = o(N/\log N)$ [3], which means that the analogue of Helson's conjecture is false in short intervals $[N, N+y]$.

6. While we were preparing a revision of this paper, further progress on Helson's problem was announced by Harper, Nikeghbali, and Radziwiłł [8]. By a completely different method, relying on Harper's lower bounds for sums of random multiplicative functions [7], these authors obtained the lower bound $\sqrt{N}(\log \log N)^{-3+o(1)}$ for both $\mathbb{E}|R_N|$ and $\mathbb{E}|S_N|$. In view of this result, it seems reasonable to conjecture that $\|S_{N,m}\|_2 / \|S_{N,m}\|_1$ is bounded whenever $m = e^{-\epsilon} \log \log N$ for $\epsilon > 0$ and that $m = \log \log N$ is the limiting case for the boundedness of this ratio. Comparing with (16) and taking into account Remark 3 above, one might wonder if the ratio $\|S_N\|_2 / \|S_N\|_1$ does indeed grow as a power of $\log \log N$.

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